UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH1202**

ASSESSMENT PATTERN : **MATH1202A**

MODULE NAME : **Algebra 2**

DATE : **08-May-08**

TIME : **14:30**

TIME ALLOWED : **2 Hours 0 Minutes**

2007/08-MATH1202A-001-EXAM-177

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*

The use of an electronic calculator is **not** *permitted in this examination.*

1. Let H be a subset of a group G. Give necessary and sufficient conditions for H to be a subgroup of G. In each of the following cases, determine if H is a subgroup of G or not, justifying your answer:

(i) $G = \mathbb{R}$ under addition, $H = \{x \in G : x \geq 0\}$;

(ii) $G = \mathbb{R}$ under addition, $H = \mathbb{Z}$;

(iii) $G = S_5$, $H = \{g \in G : g^3 = e\}$;

(iv) G is any abelian group, $H = \{g \in G : g^3 = e\}$;

(v) G is any group, K is a subgroup of G, g is an element of G, and $H = \{g^{-1}kg : k \in K\}$.

[Here S_5 denotes the permutation group on $\{1, 2, 3, 4, 5\}$ under composition.]

2. (a) State (do not prove) Lagrange's Theorem. Prove that in any finite group the order of an element divides the order of the group.

(b) Let p be a prime and $\mathbb{Z}_p^*$ the group of non-zero integers mod p under multiplication. Deduce from (a) that if $\bar{a} \in \mathbb{Z}_p^*$, then $\bar{a}^{p-1} = \bar{1}$.

(c) Find $\bar{3}^{1799}$ in $\mathbb{Z}_{19}^*$.

(d) $\bar{5}$ has order 9 in $\mathbb{Z}_{19}^*$. Show that there is no solution to $\bar{x}^3 = \bar{5}$ in $\mathbb{Z}_{19}^*$.

3. (a) Let A be an $n \times n$ matrix. Give the definition of

(i) $\det(A)$,

(ii) the (i, j)-minor of A,

(iii) the (i, j)-cofactor of A,

(iv) the adjugate, $adj(A)$ of A.

Stating any results you use, prove that $A\, adj(A) = (det A) I_n$.

(b) Let $A = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$. Find $adj(A)$ and hence find an expression for A^{-1}, stating when it is valid.

4. (a) Let A be an $n \times n$ matrix over $\mathbb{R}$. Give the definition of:

(i) an *eigenvalue* λ of A;

(ii) an *eigenvector* $\mathbf{v}$ of A;

(iii) the *eigenspace* E_λ associated to the eigenvalue λ

(iv) the *characteristic polynomial* $c_A(t)$ of A;

(v) A is *diagonalizable* (over $\mathbb{R}$).

(b) Prove that if $\lambda_1, \ldots \lambda_r$ are distinct eigenvalues of A, then the sum $\sum_{i=1}^{r} E_{\lambda_i}$ is direct; hence show that if $\sum_{i=1}^{r} \dim(E_{\lambda_i}) = n$, then A is diagonalizable.

(c) Show that the matrix $\begin{pmatrix} 3 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$

is diagonalizable.

5. Let $A = \begin{pmatrix} 1 & -5 \\ 2 & 8 \end{pmatrix}$.

(i) Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

(ii) Find A^n (for positive integers n).

(iii) Solve the system of difference equations

$$\begin{aligned} x_{n+1} &= x_n - 5y_n \\ y_{n+1} &= 2x_n + 8y_n \end{aligned}$$

for $n \geq 0$, given that $x_0 = 1$, $y_0 = 0$.

6. (a) Let A be a real symmetric matrix. Prove that any eigenvalue of A is real.

(b) Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$. Find an orthogonal matrix P such that $P^{-1}AP$ is diagonal.

END OF PAPER